ILIA NESTOROVICH VEKUA

(A Brief Review of His Scientific and Social Activities)

Ilia Nestorovich Vekua belongs to those scientists who have made a considerable contribution to the treasury of world science; He left an ineffaceable memory not only as an outstanding scientist but also as a prominent organizer of science and higher education.

I. N. Vekua was born on April 23. 1907 in the Shesheleti village. Gali district (Georgian SSR). In I925, after finishing his secondary education in the town of Zugdidi, he entered the physico-mathematical department of Tbilisi State University from which he graduated in I930. The same year he took a post-graduate course at the USSR Academy of Sciences in Leningrad. When a postgraduate student (I930-33). I. N. Vekua specialized in the equations of mathematical physics under the guidance of A. N. Krylov. N. I. Muskhelishvili and V. I. Smirnov. Using the methods of the complex variable function theory, he investigated a number of problems of static and dynamic elasticity. During that period he wrote his major works devoted to the theory of distribution of elastic waves in an infinite layer with parallel plane boundaries. These studies formed the basis of his candidate's thesis which he defended in 1937.

From the autumn of 1933 I. N. Vekua was a researcher in the physico-mathematical department of Tbilisi State University. Here, from the beginning he attracted attention as one of the most active participants of the permanent seminar on problems of mathematics and mechanics led by N. I. Muskhelishvili.

Among the scientific areas of theoretical and applied mathematics l. N. Vekua was most of all interested in the theory of elliptic partial differential equations and its various applications. The intensive research in this direction. which he began in 1936, was completed in the 1940s with the creation of an orderly theory of linear elliptic partial differential equations with analytic coefficients in the case of two independent variables. The general complex representations of solutions of such equations constructed by him proved convenient for finding new structural and qualitative properties of these solutions and for investigating a wide class of equations and boundary value problems that were difficult to investigate by the earlier methods.

A considerable part of I. N. Vekua's works in. this direction: entered his monograph „New Methods for Solving Elliptic Equations“ which was honoured with the State Prize of the USSR in 1950. In 1939 I. N. Vekua defended his thesis for a doctor's degree and in 1940 he was granted the title of professor. In 1944 I. N. Vekua was elected a corresponding member of the Academy of Sciences of the Georgian SSR, and in 1946 he became its full member.

In the period from I940 to I944 I. N. Vekua was dean of the physico-mathematical department of Tbilisi State University, from 1944 to 1947 prorector in charge of educational work, and from. 1947 to I951 academician-secretary of the Academy of Sciences of the Georgian SSR.

In the autumn of 1951 I. N. Vekua moved to Moscow where he began to work at the Central Hydroaerodynamical Institute as head of a laboratory; simultaneously, he worked at the Moscow Physico-Engineering Institute as head of the chair of theoretical mechanics. At the end of 1952 I. N. Vekua was elected professor of the chair of differential equations at Moscow M. V. Lomonosov State University, and in 1954 he was appointed deputy director of the V. A. Steklov Mathematical Institute of the Academy of Sciences of the USSR In Moscow I. N. Vekua wrote and published a large cycle of studies devoted to the so-called theory of generalized analytic functions. An attempt to construct the theory of such functions was made back in the 19th century by the Italian mathematician Beltrami. In the early 1930s Carlemann and Theoclorescu showed that a number of properties of analytic functions of one complex variable could be transferred to the solution of an elliptic system of two first order differential equations in the case of two real inde pendent variahles. I. N. Vekua created a general theory which at present forms the basis of the theory of generalized analytic functions. Using the theorems he had derived, Vekua obtained an ana 1 ytic substantiation of M. A. Lavrentyev's geometrical theory of quasiconformal mappings of plane domains, which has been recognized as one of the best achievements in the theory of functions over the last fifty years. l. N. Vekua's results in the theory of first order elliptic systems were included in his monograph “Generalized Analytic Functions” which was awarded the Lenin Prize in I963.

While in Moscow. I. N. Vekua took an active part in the ela boration oi the project on establishing a Siberian branch of the USSR Academy of Sciences. He was an active member of the spon soring group that was set up by M. A. Lavrentyev to direct the implementation of the decision of the Party and the Government on the organization of a large scientific centre in the east of the Soviet Union.

The Siberian Branch of the Academy of Sciences of the USSR was established in 1957. In March, 1958 its Presidium was elected under the chairmanship of M. A. Lavrentyev. I. N. Vekua was among the elected members of the Presidium. The same year I. N. Vekua was elected full member of the Academy of Sciences of the USSR. In 1959 Novosibirsk State University was opened at Aka demgorodok, not far from the city of Novosibirsk, and I. N. Vekua was elected its first rector (till 1965). Simultaneously he was head of the theoretical department of the Siberian Institute of Aerodynamics.

In the spring of 1965 at the request of the government of the Georgian SSR I. N. Vekua returned to his native Georgia. From 1965 to 1972 he was Rector of Tbilisi State University, and from 1972 to the end of his life President of the Academy of Sciences of the Georgian SSR.

I. N. Vekua was a warm-hearted person with high civic qualities. His endurance, courage and self-control evoked; delight and admiration. Though suffering from an incurable grave illness, the scientist persistently continued his research, resulting in the development of a new version of the mathematical theory of elastic shells. This version was included in his monograph “Theory of Shells: General Methods of Construction” which was published posthumously in Moscow in 1932. The monograph was awarded the State Prize of the USSR and its English translation was published by Pitman Publishers in 1985.

I. N. Vekua's scientific contribution earned him international recognition. He was a member of foreign Academies: the Academy of Sciences of the German Democratic Republic (Berlin), the Academy of Natural Sciences “Leopoldina” (Halle), the Academy of Sciences of Literature and Art in Palermo (Sicilian Academy of Sciences), the Polish Society of Theoretical and Applied Mechanics, the Danish Centre of Applied Mathematics and Mechanics, and of other scientific societies. He was a member of the General Assembly of the International Union on Theoretical and Applied Mechanics (IUTAM). He was granted the titles of honorary doctor of Halle University and honorary senator of Jena University.

The Party and the Government highly appreciated I. N. Vekua's services. Among numerous government awards he was decorated with the Gold Star of the Hero of Socialist Labour and five orders of Lenin.

I. N. Vekua died on September 2, 1977. The grateful Georgian people buried him at the pantheon of the celebrated sons of the Georgian land, on Mount Mtatsminda. Now we shall give a brief account of the most typical features of l. N. Vekua's rich legacy that have greatly influenced the development of the respective problems of mathematics. The general linear boundary value problem for analytic functions of one complex variable, studied comprehensively by I. N.Vekua, holds the key position in the modern theory of the so-called non-Fredholmian problems for elliptic equations.

One of the basic problems of the function theory is associated with the names of Riemann and Hilbert: Define a function analytic in D and satisfying the boundary condition $\Phi(z)$ analityc in D and satisfying the boundary condition

$$Re[\lambda(t)\Phi + (t)] = g(t), t \in \partial D, \quad (1)$$

where λ and g are the known functions at the boundary δD of D, while $\Phi^+(t)$ is the boundary value of the unknown function for z → t from D.

By introducing the notation

$$\lambda(t) = \lambda_1(t) + i\lambda_2(t), \Phi(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y},$$

problem (I) can be reduced to the problem with a directional derivative, i. e., to the Poincare problem: Define a function harmonicin D and satisfying the following boundary condition

$$\lambda_1\frac{\partial u}{\partial x} + \lambda_2\frac{\partial u}{\partial y} = g(t), t = x + iy \in \partial D. \quad (2)$$

in connection with the study of problems (1) and (2) the one-dimensional singular integral equation

$$A\varphi(t) = \alpha(t)\varphi(t) + \beta(t)S\varphi(t) + T\varphi(t) = f(t), t \in \partial D, \quad (3)$$

is considered, where S is a Cauchy-type singular operator.

$$S\varphi(t) = \frac{1}{\pi i}\int_{\partial D}\frac{\varphi(\tau)d\tau}{\tau - t}, t \in \partial D,$$

and T is the Fredholm integral operator.

One of the main questions of the theory of equations of form (3) is the reduction of this equation to an equivalent second kind Fredholm equation (the problem of equivalent regularization). I. N. Vekua's solution of this problem is considered a brilliant result in the theory of singular integral equations. Developing Carlemann's idea, in the l940s I. N. Vekua worked out—in classical assumptions - a way for constructing a theory of integral equations (3) known at present as the Carlemann-Vekua method. it involves three stages: 1) solutions of a characteristic singular integral equation (i. e., eq. (8) for T=O) as well as of its associated equationare constructed eilectively; 2) these solutions are used for an equivalent regularization of eq. (3) and of its associated equation; 3) Fredholm integral equations, constructed in this manner, are used for proving the Noether theorems for eq. (3).

The regularization problem can be solved using, in addition to the above-mentioned method, the method of regularization by multiplication of operators. The idea of this method is in the tollowing: It is required to construct an operator B of type A (See(3)) such that the equation BAφ=Bf be Fredholmian. ln this case B is said to be the left regularizer ol A. It the equations Aφ=f and BAφ=Bf are equivalent, whatever f from the considered class of functions is, B is said to be the left equivalent regularizer of A. It was known that such an operator does not always exist. ln this connection a question arises: Is it possible to formulate the problem of constructing a Fredholm equation equivalent to eq. (3) in such a way that it should always have a solution? I. N. Vekua answered the question positively. He showed that there exists an operator B, constructed effectively in quadratures, such that either the equations Aφ=f and BAφ=Bf or A=φf and ABφ=f are equivalent in the sense that if either of the equations is solvable, then so is the other, and the connection φ=Bφ exists between their solutions.

Using his theory of eq. (3), I. N. Vekua succecded in solving completely the Riemann-Hilbert problem (1) in the following general formulation: In the domain D whose boundary δD is a sufficiently smooth, simple, closed curve it is required to find an analytic function Φ satisfying the boundary conditions

$$Re\sum_{k=0}^{m}\{\lambda_k(t)[\Phi^{(k)}(t)] + T_k[(\Phi^{(k)}) +]\} = f(t),\ t \in \partial D, \quad (4)$$

where$(\Phi^{(\kappa)})^{+}$ is a boundary value of κthe order derivative of Φ from D; λ_κ and f are the functions given on δD and T_κ is a Fredholm integral operator.

The integral representation of an analytic function, bearing I. N. Vekua's name, has played an important part in these investigations.

I. N. Vekua's results obtained in connection with problem (4) formed the basis of his further research devoted to constructing a theory of normally solvable boundary value problems in the case of the following second order elliptic differential equation

$$\Delta u + a_1(x,y)\frac{\partial u}{\partial x} + a_2(x,y)\frac{\partial u}{\partial x} + a_3(x,y)u = 0, \qquad (5)$$

where α_1, α_2, α_3 are analytic functions. These problems are an essential generalization of the Poincare boundary value problem (2) in the case of eq. (5). Indeed, the boundary condition is of the form

$$\sum_{j+k\le m} a_{jk}(t)\left[\frac{\partial^{j+k}u}{\partial x^j \partial y^k} + T_{jk}\left(\frac{\partial^{j+k}u}{\partial x^j \partial y^k}\right)\right] = f(t), t \in \partial D, \qquad (6)$$

where $\alpha_{J\kappa}$, and f are the known real functions defined on δD, $T_{J\kappa}$ are the Fredholm integral operators.

In constructing the theory of this problem I. N. Vekua made use of the integral representation of all regular solutions 0f eq. (5)

$$u(x,y) = Re\left[\alpha(z,\bar{z})\varphi(z) + \int_0^z \beta(z,\bar{z},t)\varphi(t)dt\right], \qquad (7)$$

where φ is an arbitrary analytic function, while α and β are functions constructed by means of the coefficients α_1, α_2, and α_3.

Formulas similar to (7) were obtained by T. Carlemann, G.Lewy and S. Bergman. The method of constructing formula (7), known as the Riemann-Vekua method, is considered to be the most simple, clear and constructive one.

I. N. Vekua generalized formula (7) for the elliptic equation

$$\Delta^n u + \sum_{k=1}^{n} L_k\left(\Delta^{n-k}u\right) = 0, \qquad (8)$$

where L_κ order differential operator with analytic coefficients.

Employing his formulas. I. N. Vekua investigated the first boundary value problem: Find the regular solution of this equation in the simply connected domain D, satisfying the conditions

$$\left.\frac{d^j u}{dv^j}\right|_{\partial D} = f_j, j = 0,1,\dots,n-1,$$

where ν is the outward normal of δD,

In the theory of general complex representations of solutions of elliptic equations I. N. Vekua discovered a remarkable fact on a possibility‘ of equivalent reduction of any boundary value problem for eq. (8) to the corresponding boundary value problem for a system of analytic functions.

As is well-known, the theory of analytic functions φ(z)=u (x, y)+ iυ (x, y) of one complex variable z=x+iy is the theoryof the Cauchy-Riemann system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

This system is the particular case of the elliptic system.

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + au + bv = 0,$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + cu = dv = 0, \tag{9}$$

with the real coefficients a, b, c, d which are functions of the real variables x, y.

Introducing the notations $W(z) = u + iv, 2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, 4A = a + d + i(c - b), 4B = a - d + i(c + b)$, system (9) can be rewritten as

$$\frac{\partial W}{\partial \bar{z}} + AW + B\overline{W} = 0 \tag{10}$$

Back in the 19th century Beltrami and Picard tried to construct a theory of generalized analytic functions w=u+iv. Important results in this direction were obtained by T. Carlemann and M. A. Lavrentyev. I. N. Vekua's intensive research also yielded basic results that formed the principal component part of the modern theory of functions satisfying the equation

$$\frac{\partial F}{\partial \bar{z}} - q_1\frac{\partial F}{\partial z} - q_2\frac{\partial \bar{F}}{\partial \bar{z}} + AF + B\bar{F} = 0 \tag{11}$$

If $|q_1| + |q_2| < 1$, then eq, (11) is a complex form of a linear elliptic system of two equations with respect to the real and imaginary parts of F. In the constructed theory the known functions.
q_1, q_2, A, and B are required to be sufficiently smooth.

I. N. Vekua made a substantial contribution to the theory of metaharmonic functions constituting solutions of the Helmholtz equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_p^2} + \lambda^2 u = 0, \lambda = const, p \geq 2 \tag{12}$$

He gave the following integral representation of metaharmonic functions

$$u(x_1, x_2, \dots, x_p) = u_0(x_1, \dots, x_p) -$$

$$-\int_0^1 u_0(x_1 t, \dots, x_p t) t^q \frac{\partial}{\partial t} I_0(\lambda r\sqrt{1-t}) dt,$$

where u_0 is an arbitrary harmonic function, $q = \frac{p-2}{2}, r^2 = x_1^2 + \cdots + x_p^2,\ I_0$ and I_0 is a Bessel function. He also constructed an inverse integral representation of this formula and, moreover, in the case of a real λ he showed that under the Sommerfeld conditions

$$\frac{\partial u}{\partial r} - i\lambda u = 0(r^{-q-1/2}), u = O(r^{-q-1/2})$$

providing the uniqueness of the solution of the external Dirichlet problem in the case of eq. (12), the second condition was the con sequence of the first one.

I. N. Vekua showed that his method of constructing solutions of elliptic linear equations could be used in the investigation of the properties of solutions of some nonlinear elliptic equations. Thu, for example, he studied the properties of solutions of the Gaussian equation

$$\Delta \log \upsilon(x, y) = -2K(x, y)\, \upsilon(x, y)$$

which enabled him to find a simple proof of the well—known Hilbert theorem on the nonexistence of a regular surface with a negative curvature, conformally homeomorphic to the whole plane.

The range of theoretical results obtained by I. N. Vekua is wide. On the basis of his methods of investigation of elliptic equations an orderly theory of elastic shells was constructed. ln particular, I. N. Vekua proposed two versions of this theory. one of which is used in the investigation of thin sloping shells, and the other in the construction of a membrane theory of shells.

In the membrane theory of shells with an alternating Gaussian curvature the main part is played by an equation of the mixed type, in particular, by the Holgren-Hellerstedt equation

$$y^{2m+1}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

($m \geq 0$ is an integer), and the Lavrentyev-Bitsadze equation

$$\frac{\partial^2 u}{\partial x^2} + \operatorname{sign} y\, \frac{\partial^2 u}{\partial y^2} = 0,$$

for which the Tricomi problem and its various generalizations acquire a clearly defined mechanical sense. This fact, discovered by I. N. Vekua in the mid-l950s evoked great interest among mathematicians and mechanicians, since prior to that period equation of mixed type were applied only to problems of aerodynamics.

Because of the depth and importance of I. N. Vekua's investigations and the new scientific trends founded by him he ranks among the most outstanding scientists of our time.

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